Some Iterative methods for a Mixed Equilibrium Problem

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Abstract: In this paper, using fixed point and resolvent equation techniques, we suggest some iterative algorithms for a mixed equilibrium problem. Further we define θ-mixed psudomonotonicity and mixed psudocontractive for a bifunction which extend the concepts of psudomonotonicity and psudocontractive for a mapping. Furthermore, using these concepts, we discuss the convergence analysis of these algorithms.

Key words: Mixed equilibrium problem, iterative algorithms, fixed point, resolvent equation technique, Ө-mixed psudomonotonicity and mixed psudocontractive bifunctions.

Introduction

The theory of equilibrium problems has emerged as an interesting branch of applied mathematics, permitting the general and unified study of a large number of problems arising in mathematical economics, optimization and operations research.

In 1999 Moudafi and Thera [9] introduced a class of mixed equilibrium problems and discussed several numerical methods including an auxiliary problem principle, a selection method as well as a dynamical procedure to solve the mixed equilibrium problems. Further, in 2002, Moudafi [8] discussed some resolvent methods for solving mixed equilibrium problems which required the monotonicity and Lipschitz continuity of the mapping.

Motivated by recent work going in this direction, we consider a mixed equilibrium problem (in short MEP) in Hilbert space. We suggest some splitting type iterative methods by modifying the resolvent methods in the spirit of the extra-gradient methods for solving MEP. These new iterative methods differ from the known resolvent methods [11-13,15-16]. Using the resolvent operator technique, we establish the equivalence between mixed equilibrium problems, fixed point problems and the resolvent equations. These alternative formulations are used to suggest and analyze a number of iterative methods for solving mixed equilibrium problem. Further we extend the relaxation technique developed by He [2,3], Konnov [4,5] and Noor [14] to show that the convergence of the proposed iterative methods requires either only mixed monotonicity or pseudomonotonicity, so our resolvent represent a significant improvement than the previously known results. In brief, our results can be viewed as an extension of the results of Koperlevich [6], Lions and Mercier [7], Passty [18] Solodov and Tseng [19], Sun [20-21], Noor and Rassias [17] and Noor [10-16] for solving variational inequalities and complementarity problems.

In section 2, we consider a mixed equilibrium problem (MEP) and recall some concepts and results which are essential for the presentation of the results of this paper.

In section 3, we suggest some iterative algorithm for MEP and show that the approximate solution obtained from iterative algorithm is strongly convergent to the exact solution of MEP, which requires only mixed monotonicity of the operator, while section 4 deals with convergence analysis of some algorithms based on resolvent equations for MEP, which requires the Θ-mixed pseudomonotonicity and mixed pseudocontractive of the operator.

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Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $|.|$, respectively. Let K be nonempty, closed, convex set in H; $T,S:K\to K$ be nonlinear mappings, and $N:K\times K\to K$ be a nonlinear mapping. If $F:K\times K\to R$ is a given bifunction satisfies $F(x,x)=0$, $\forall x\in K$, then we consider the following *mixed equilibrium problem* (for short, MEP): Find $x\in K$ such that

$$F(x,y)+\langle N(Tx,Ax),y-x\rangle\geq 0 \quad \forall y\in K. \quad (2.1)$$

This problem generalizes MEP studied in [9] and has potential and useful applications in nonlinear and mathematical economics, see [8,9].

The following definitions and theorem will be needed in the sequel.

Definition 2.1 [8]. Let $F:K\times K\to R$ be a real-valued function. Then F is said to be:

(a) *monotone* if $F(x,y)+F(y,x)\leq 0$, for each $x,y\in K$;

(b) *strictly monotone* if $F(x,y)+F(y,x)<0$, for each $x,y\in K$, with $x\neq y$;

(c) *upper-hemicontinuous*, if for all $x,y,z\in K$, $\limsup_{t\to 0} F(tz+(1-t)x, y)\leq F(x,y)$.

Theorem 2.1 [8] If the following conditions hold true for $F:K\times K\to R$:

(i) F is monotone and upper-hemicontinuous;

(ii) $F(x,.)$ is convex and lower semicontinuous for each $x\in K$;

(iii) there exists a compact subset B of R^n and there exists $y_0\in B\cap K$ such that $F(x,y_0)<0$ for each $x\in K\backslash B$, then the set of solutions to the following equilibrium problem:

Find $x\in K$ such that $F(x,y)\geq 0$, $\forall y\in K$, is nonempty convex and compact. Moreover, if F is strictly monotone, then the solution of equilibrium problem is unique.

Let us recall the extension of the Yosida approximation notion introduced in [5]. Let $\mu>0$, for a given bifunction F, the associated *Yosida approximation*, F_μ, over K and the corresponding *regularized operator*, A^F_μ, are defined as follows:

$$F_\mu(x,y)=\langle \frac{1}{\mu}(x-J^F_\mu(x)),y-x\rangle \text{ and } A^F_\mu:=\frac{1}{\mu}\left(x-J^F_\mu(x)\right),$$

in which $J^F_\mu(x)\in K$ is the unique solution of

$$\mu F(J^F_\mu(x),y)+\langle J^F_\mu(x)-x,y-J^F_\mu(x)\rangle\geq 0, \quad \forall y\in K. \quad (2.2)$$

Remark 2.1 [8]

(i) The existence and uniqueness of the solution of problem (2.2) follows by invoking Theorem 2.1.

(ii) If $F(x,y)=\sup_{u\in Bx}\langle u, y-x\rangle$ and $K=H$, B being a maximal monotone operator, it directly yields

$$J^F_\mu(x)=(I+\mu B)^{-1}x \text{ and } A^F_\mu(x)=B_\mu(x),$$

where $B_\mu:=\frac{1}{\mu}\left(I-(I+\mu B)^{-1}\right)$ is the Yosida approximation of B, and one recover classical concepts.

(iii) The operator J^F_μ is co-coercive and nonexpansive.

In the sequel, we assume that the bifunction F satisfies conditions of Theorem 2.1.

Iterative algorithms and convergence analysis

First we define the following concepts.

Definition 3.1. Let $T,A:K\to K$, $F:K\times K\to R$ and $N:K\times K\to K$ be nonlinear mappings. Then, for all $x,y,z\in K$, N is said to be:

(a)[9] *mixed monotone* with respect to T and A, if

$$\langle N(Tx,Ax)-N(Ty,Ay),x-y\rangle\geq 0;$$

(b) θ-*mixed pseudomonotone* with respect to T and A, where θ is a real-valued multivariate function, if

$\langle N(Tx,Ax),z-x\rangle+\theta\geq 0$ implies $\langle N(Tz,Az)-,z-x\rangle+\theta\geq 0$;

(c) δ-*mixed psudocontractive* with respect to T and A, if

$\langle N(Tx,Ax)-N(Ty,Ay),x-y\rangle\leq\delta \lVert x-y\rVert^2$

We remark that the concepts defined above are the natural generalization of their corresponding usual concepts.

Lemma 3.1. MEP (2.1) has a solution x if and only if x satisfies the equation

$$x=J_{\mu}^{F}(x-\mu N(Tx,Ax)),\ \text{for}\ \ \mu>0.(3.1)$$

We now define the residue vector $R(x)$ by the relation

$$R(x)=x-J_{\mu}^{F}[x-\mu N(Tx,Ax)].$$

Invoking Lemma 3.1, one can observe that $x\in K$ is a solution of MEP (2.1) if and only if $x\in K$ is a zero of the equation

$$R(x)=0.$$

The fixed point formulation given in Lemma 3.1 for MEP (2.1) is very useful from the numerical point of views. This fixed point formulation enables us to suggest and analyze the following iterative algorithm.

Algorithm 3.1 For a given $x_0\in H$, compute the approximate solution x_{n+1}, by the iterative scheme

$$x_{n+1}=J_{\mu}^{F}[x_n-\mu N(Tx_n,Ax_n)],\ n=0,1,2,...$$

Rewrite the equation (3.1) in the form

$$x=J_{\mu}^{F}[x-\mu N(TJ_{\mu}^{F}[x-\mu N(Tx,Ax)],AJ_{\mu}^{F}[x-\mu N(Tx,Ax)])]$$

by updating the solution. This fixed point formulation allows us to suggest the following extraresolvent method.

Algorithm 3.2 For a given $x_0\in H$, compute x_{n+1}, by the iterative scheme

$$x_{n+1}=J_{\mu}^{F}[x_n-\mu N(TJ_{\mu}^{F}[x_n-\mu N(Tx_n,Ax_n)],AJ_{\mu}^{F}[x_n-\mu N(Tx_n,Ax_n)])],\ n=0,1,2...$$

If $F(x,y)=\delta_K(y)-\delta_K(x)$, for all $x,y\in K$, then $J_{\mu}^{F}=P_K$, the projection of H onto K, then Algorithm 3.2 reduces the extragradient method of Korpelevich [6].

Now define the residue vector $R(x)$ by the relation

$$R(x)=x-J_{\mu}^{F}[x-\mu N(TJ_{\mu}^{F}[x-\mu N(Tx,Ax)],AJ_{\mu}^{F}[x-\mu N(Tx,Ax)])].$$

We can easily observe that $x\in K$ is a solution of MEP (2.1) if and only if $x\in K$ is a zero of the equation

$$R(x)=0.(3.2)$$

For a constant $\gamma\in(0,2)$, (3.2) can be written as

$$x+\mu N(Tx,Ax)=x+\mu N(Tx,Ax)-\gamma R(x).$$

This formulation is used to suggest a new implicit method for solving MEP (2.1).

Algorithm 3.3 For a given $x_0\in H$, compute x_{n+1}, by the iterative scheme

$$x_{n+1}=x_n+\mu N(Tx_n,Ax_n)-\mu N(Tx_{n+1},Ax_{n+1})-\gamma R(x_n),\ n=0,1,2...,(3.3)$$

If $\gamma=1$, then Algorithm 3.3 reduces to:

Algorithm 3.4 For a given $x_0\in H$, compute x_{n+1}, by the iterative scheme

$$x_{n+1}=(1+\mu N(T(.),A(.)))^{-1}[J_{\mu}^{F}[I+\mu N(T(.),A(.))]+\mu N(T(.),A(.))]x_n,\ n=0,1,2...,$$

where $(N(T(.),A(.)))x=N(Tx,Ax)$, $\forall x\in K$, which is a variant of the Douglas-Rachford [1] splitting algorithm studied by Lions and Mercier [7], and appears to be new for MEP (2.1).

Now, we prove the following results.

Theorem 3.1 Let $\bar{x}\in K$ be a solution of MEP (2.1). If N is mixed monotone with respect to T and A, then

$$\langle x-\bar{x}+\mu[N(Tx,Ax)-N(T\bar{x},A\bar{x})],R(x)\rangle\geq|R(x)|^2\ \forall x\in K.\quad(3.4)$$

Proof Let $\bar{x}\in K$ be a solution of MEP (2.1), then

$$F(\bar{x},y)+\langle N(T\bar{x},A\bar{x}),y-\bar{x}\rangle\geq 0\ \ \forall y\in K.\quad(3.5)$$

Taking $y=x-R(x)$ in (3.4), we have

$$\mu F(\bar{x},x-R(x))+\langle\mu N(T\bar{x},A\bar{x}),x-R(x)-\bar{x}\rangle\geq 0.\quad(3.6)$$

Setting: $y:=\bar{x}$, $z=J_\mu^F(x):=x-R(x)$ and $x:=x-\mu N(Tx,Ax)$ in (2.2), we have

$$\mu F(x-R(x),\bar{x})+\langle x-R(x)-(x-\mu N(Tx,Ax)),\bar{x}-(x-R(x))\rangle\geq 0,$$

i.e. $$\mu F(x-R(x),\bar{x})+\langle R(x)-\mu N(Tx,Ax),x-R(x)-\bar{x}\rangle\geq 0.\quad(3.7)$$

Adding (3.6) and (3.7), we have

$$\mu[F(x-R(x),\bar{x})+F(\bar{x},x-R(x)]+\langle\mu N(T\bar{x},A\bar{x})-\mu N(Tx,Ax)+R(x),x-R(x)-\bar{x}\rangle\geq 0.\quad(3.8)$$

Since F is monotone, (3.8) implies that

$$\langle\mu N(T\bar{x},A\bar{x})-\mu N(Tx,Ax)-R(x),\bar{x}-(x-R(x))\rangle\geq 0,$$

$$\langle R(x)-\mu[N(Tx,Ax)-N(T\bar{x},A\bar{x})],x-\bar{x}-R(x)\rangle\geq 0.\quad(3.9)$$

Since N is mixed monotone with respect to T and A, from (3.9), we have

$$\langle x-\bar{x}-\mu[N(Tx,Ax)-N(T\bar{x},A\bar{x})],R(x)\rangle$$

$$=\langle R(x),R(x)\rangle+\langle R(x)-\mu[N(Tx,Ax)-N(T\bar{x},A\bar{x})],x-\bar{x}-R(x)\rangle+\mu\langle N(Tx,Ax)-N(T\bar{x},A\bar{x}),x-\bar{x}\rangle\geq|R(x)|^2.$$

Theorem 3.2 Let $\bar{x}\in K$ be the solution of MEP (2.1) and x_{n+1} be the approximate solution obtained from Algorithm 3.3, then

$$|x_{n+1}-\bar{x}+\mu[N(Tx_{n+1},Ax_{n+1})-N(T\bar{x},A\bar{x})]|^2$$

$$\leq|x_{n+1}-\bar{x}+\mu[N(Tx_n,Ax_n)-N(T\bar{x},A\bar{x})]|-\gamma(2-\gamma)|R(x_n)|^2\quad(3.10)$$

Proof Since $\bar{x}$ is a solution of MEP (2.1) and x_{n+1} satisfies (3.3), then using Theorem 3.1, we have

$$|x_{n+1}-\bar{x}+\mu[N(Tx_{n+1},Ax_{n+1})-N(T\bar{x},A\bar{x})]|^2$$

$$=|x_n-\bar{x}+\mu[N(Tx_n,Ax_n)-N(T\bar{x},A\bar{x})]-\gamma R(x_n)|^2$$

$$\leq|x_n-\bar{x}+\mu[N(Tx_n,Ax_n)-N(T\bar{x},A\bar{x})]|^2-2\gamma|R(x_n)|^2+\upsilon^2|R(x_n)|^2$$

$$=|x_n-\bar{x}+\mu[N(Tx_n,Ax_n)-N(T\bar{x},A\bar{x})]|^2-\gamma(2-\gamma)|R(x_n)|^2.$$

Next, we prove that approximate solution obtained from Algorithm 3.3 converges strongly to a solution of MEP (2.1).

Theorem 3.3 Let H be a finite dimensional space. The approximate solution x_{n+1} obtained from Algorithm 3.3 converges to a solution $\bar{x}$ of MEP (2.1).

Proof Let $\bar{x}\in K$ be the solution of MEP (2.1). From (4.7), it follows that the sequence $\{x_n\}$ is bounded and

$$\sum_{n=0}^{\infty}\gamma(2-\gamma)|R(x_n)|^2\leq|x_0-\bar{x}+\mu[N(Tx_0,Ax_0)-N(T\bar{x},A\bar{x})]|^2,$$

and consequently

$$\lim_{n\to\infty}R(x_n)=0.$$

Let $\hat{x}$ be a limit point of $\{x_n\}$. A subsequence $\{x_{n_i}\}$ of $\{x_n\}$, which converges to $\hat{x}$. Since $R(x)$ is continuous, so

$$R(\hat{x})=\lim_{i\to\infty}R(x_{n_i})=0,$$

and hence $\hat{x}$ is the solution of MEP (2.1) and

$$|x_{n+1}-\hat{x}+\mu[N(Tx_{n+1},Ax_{n+1})-N(T\hat{x},A\hat{x})]|^2$$

$$\leq|x_n-\hat{x}+\mu[N(Tx_n,Ax_n)-N(T\hat{x},A\hat{x})]|^2.$$

It follows that the sequence $\{x_n\}$ has exactly one limit point and $\lim_{n\to\infty}x_n=\hat{x}\in K$, satisfies the MEP (2.1).

4. Resolvent equation technique

Now related to MEP (2.1), we consider the following Resolvent equation (for short, RE): Find $z\in H$ such that for $x\in K$,

$N(Tx,Ax)+A_\mu^F(z)=0$,(4.1)

And $x=J_\mu^F(z)$, for $\mu>0$.(4.2)

Lemma 4.1. MEP (2.1) has a solution x if and only if RE (4.1)-(4.2) has a solution $z\in H$ where

$x=J_\mu^F(z)$(4.3) and

$z=x-\mu N(TJ_\mu^F[x-\mu N(Tx,Ax)],AJ_\mu^F[x-\mu N(Tx,Ax)])$, for $\mu>0$.(4.4)

Lemma 4.1 shows that MEP (2.1) and RE (4.1)-(4.2) both have the same solution set. Based on Lemma 4.1, we suggest a new iterative algorithm for MEP (2.1).

Using the fact that $A_\mu^F=\frac{1}{\mu}(I-J_\mu^F)$, RE (4.1)-(4.2) can be written as $z-J_\mu^F(z)+\rho N(TJ_\mu^F(z),AJ_\mu^F(z))=0$.

For a step size γ, we can write above equation as

$x=x-\gamma[z-J_\mu^F(z)+\rho N(TJ_\mu^F(z),AJ_\mu^F(z))]=0$.

This fixed point formulation allows us to suggest the following iterative algorithm for MEP (2.1).

Algorithm 4.1 For a given $x_0\in K$, compute the approximate solution x_{n+1} by the iterative schemes

$$z_n=x_n-\mu N(TJ_\mu^F[x_n-\mu N(Tx_n,Ax_n)],AJ_\mu^F[x_n-\mu N(Tx_n,Ax_n)])]$$

$$w_n=z_n-J_\mu^F z_n+\mu N(TJ_\mu^F z_n,AJ_\mu^F z_n))$$

$$x_{n+1}=x_n-\gamma w_n,\ n=0,1,2,...$$

Theorem 4.1 Let $\hat{x}\in K$ be the solution of MEP (2.1). N is θ-mixed pseudomonotone with respect to T and A where $\theta(x,y)=F(x,y)\ \forall x,y\in K$ and δ-mixed pseudocontractive with respect to T and A. Then

$\langle x-\hat{x},R(x)-\mu[N(Tx,Ax)+\mu[N(Tz,Az)\rangle\geq(1-\mu\delta)|R(x)|^2\ \forall x\in K$(4.5)

where $z:=J_\mu^F[x-\mu N(TJ_\mu^F[x-\mu N(Tx,Ax)],AJ_\mu^F[x-\mu N(Tx,Ax)])]$.

Since N is θ-mixed pseudomonotone with respect to T and A, where $\theta(x,y)=F(x,y)\ \forall x,y\in K$, then for all $x,\hat{x}\in K$,

$\langle N(T\hat{x},A\hat{x}),z-\hat{x}\rangle+\theta\geq0$

implies

$$\langle N(Tz,Az),z-\hat{x}\rangle+\theta\geq0$$

$$\langle N(Tz,Az),z-\hat{x}\rangle\geq-F(\hat{x},z)$$

$$\geq F(z,\hat{x})$$

i.e., $-F(z,\hat{x})+\langle N(Tz,Az),z-\hat{x}\rangle\geq0$

i.e.,for $\mu>0$,

$$-\mu F(x-R(x),\hat{x})+\mu\langle N(T(x-R(x)),A(x-R(x))),x-R(x)-\hat{x}\rangle\geq0 \quad (4.6)$$

Adding (4.6) and (3.7), we have

$$\langle R(x)-\mu N(Tx,Ax)+\mu N(T(x-R(x)),A(x-R(x))),x-\hat{x}\rangle$$

$$\geq\langle R(x)-\mu N(Tx,Ax)+\mu N(T(x-R(x)),A(x-R(x))),R(x)\rangle$$

$$\geq|R(x)|-\mu N(Tx,Ax)+N(T(x-R(x)),A(x-R(x))),(x-R(x)\rangle$$

Since N is mixed pseudocontractive with respect to T and A, above inequality implies

$$\langle R(x)-\mu N(Tx,Ax)+\mu N(T(x-R(x)),A(x-R(x))),x-\bar{x}\rangle$$

$$\geq(1-\mu\delta)|R(x)|^2.$$

The following theorem is immediately followed by Theorem 4.1 and Algorithm 4.1.

Theorem 4.2 The sequence $\{x_n\}$ is generated by Algorithm 4.1 for MEP (2.1) satisfies the inequality

$$|x_{n+1}-\bar{x}|^2\leq|u_n-\bar{u}|^2-\gamma(2-\gamma-2\mu\delta)|R(x_n)|^2 \quad \forall x\in K,$$

where ū is a solution of MEP (2.1).

We remark that following the technique of Theorem 3.3, we can easily show that the approximate solution x_{n+1} obtained from Algorithm 4.1 converges to the exact solution of $\bar{x}\in K$ of MEP (2.1).

We remark that following the technique of Theorem 3.3, we can easily show that the approximate solution x_{n+1} obtained from Algorithm 4.1 converges to the exact solution of x 2 K of MEP (2.1).

References

[1] J. Douglas and H.H. Rachford, on the numerical solutions of the heat condition problem in 2 and 3 space variables, Trans. Amer. Math. Soc., 82(1956) 421-439.

[2] B. He, A class of projection and contraction methods for monotone variational inequalities, Appl. Math. Optim., 35(1997) 69-76.

[3] B., He, A class of new methods for monotone variational inequalities, Preprint(1995), Institute of Mathematics, Nanjing University, Nanjing, China,

[4] I. V. Konnov, Combined relaxation methods for nding equilibrium points and solving related problems, Russian Math. (Iz. Vuz), 37(2)(1993) 44-51.

[5] I. V. Konnov, A class of combined iterative methods for solving variational in-equalities, J. Optim. Theory Appl., 94(1997) 677-693.

[6] G. M. Korpelevich, Extregradient methods for nding saddle points and other problems, Matecon, 12(1976) 747-756.

[8] P. L. Lions and B. Mercier, Splitting algorithms for the sum of two monotone operators, SIAM J. Num. Anal., 16(1979) 964-969.A. Mouda , Mixed equilibrium problems: Sensitivity analysis and algorithmic aspect, Comput. Math. Appl., 44(2002) 1099-1108.

[9] A. Mouda and M. Thera, Proximal and dynamical approachs to eqilibrium problems, Lecture Notes in Economics and Mathematical Systems, Vol. 477, pp. 187-201, Springer, 1999.

[10] M. A. Noor, Iterative schemes for quasimonotone mixed variational inequalities, Optimization, 50(2001) 29-44.

[11] M. A. Noor, Algorithms for general monotone mixed variational inequalities, J. Math. Anal. Appl., 229(1999) 330-343.

[12] M. A. Noor, Generalized monotone mixed variational inequalities, Math. Com-puter Modelling, 29(1993) 87-93.

[13] M. A. Noor, A new iterative method for monotone mixed variational inequalities, Math. Computer Modelling, 26(7)(1997) 29-34.

[14] M. A. Noor, An implicit method for mixed variational inequalities, Appl. Math. Letters, 11(4)(1998) 109-113.

[15] M. A. Noor, An extraresolvent method for monotone mixed variational inequal-ities, Math. Computer Modelling, 29(1999), 95-100.

[16] M. A. Noor, Some algorithms for general monotone mixed variational

inequalities, Math. Computer Modelling, 29(1999), 1-9.
[17] M.A. Noor and Th. M. Rassias, Projection methods for monotone variational inequalities, J. Math. Anal. Appl., 237(1999) 405-412.
[18] G. B. Passty, Ergodic convergence to zero of the sum of two monotone operators in Hilbert space, J. Math. Anal. Appl., 72(1979) 383-390.
[19] M. V. Solodov and P. Tseng, Modi ed projection-type methods for monotone variational inequalities, SIAM J. Control. Optim., 34(5)(1996) 1814-1836.
[20] D. Sun, A class of iterative methods for solving nonlinear projection equations, J. Optim. Theory Appl., 91(1996) 123-140.
[21] D. Sun, A projection and contraction method for the nonlinear complementarity problem and its extensions, Chin. J. Num. Math. Appl., 16(1994) 73-84.
[22] L.C. Zeng, S. Schaible and J.-C. Yao, Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities, J. Optim. Theory Appl. 124(2005) 725-738.